

Bounds for B-coloring planar and outerplanar graphs

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ABSTRACT

A proper coloring of the edges of a graph is one in which every $K_{1,2}$ is totally multicolored. A B-coloring is a coloring of the edges of G in which every $K_{1,2}$ and every $K_{2,2}$ is totally multicolored.

In this paper, we improve two results by the authors and Gyárfás. The first shows that an outerplanar graph with maximum degree Δ can be B-colored with $\max\{\Delta, 9\}$ colors, improving a bound of $\Delta + 1$ and this is best possible up to the constant 9.

The second shows that a planar graph with maximum degree Δ can be B-colored with $\max\{2\Delta + 4, 22\}$, improving a bound of $2\Delta + 8$.

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1. Introduction

A *planar graph* is a graph that can be drawn on the Euclidean plane such that any two edges can only intersect at their endvertices. An *outerplanar graph* is a planar graph that can be drawn such that all of its vertices lie on the boundary of the same face. In this paper, all of our graphs will be simple. That is, they will have no loops or multiple edges.

An edge coloring of a graph G is *proper* if adjacent edges must receive different colors. In this case, the edges incident with the same vertex are said to be *totally multicolored* or *rainbow*. Equivalently, the edges of any copy of the complete bipartite graph $K_{1,2}$ must be different. Vizing's Theorem [11] says that any graph G with maximum degree Δ can be properly colored with either Δ or $\Delta + 1$ colors. Holyer [9] proved that it is NP-hard to determine whether Δ colors suffice for an arbitrary graph G .

The $(7, 4)$ -problem of Brown, Erdős and Sós [3] claims that any triple system (i.e., 3-uniform hypergraph) on n vertices such that no set of 7 vertices has 4 triples has a total of $o(n^2)$ triples.

Closely related to the $(7, 4)$ -problem, Gyárfás and Sárközy [8] introduced and studied A- and B-colorings: an A-coloring is a proper coloring where the union of any two color classes does not contain paths or cycles with four edges and a B-coloring is a proper coloring where every 4-cycle of G must be rainbow. A-colorings are also called *star-edge colorings* in the literature and were defined in [5] (they appeared also in [6]) and they have been extensively studied. See, for example, [1, 4, 12], or the survey [10]. We define $q_A(G)$ [$q_B(G)$] to be the minimum number of colors sufficient for an A-coloring [B-coloring] of the graph G .

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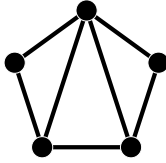
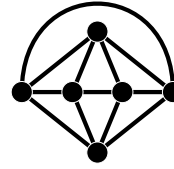
((A)) Outerplanar graph with $\Delta = 4$ but $\chi_B = 5$.((B)) Planar graph with $\Delta = 4$ but $\chi_B = 12$.

Fig. 1. Small examples.

In [7], $q_B(G)$ was studied for planar graphs. After proving some partial results, Gyárfás and the present authors formulated the following conjecture.

Conjecture 1 ([7]). *There is a Δ_0 such that for $\Delta \geq \Delta_0$ we have $q_B(G) \leq 2\Delta$ for any planar G with maximum degree $\Delta(G) = \Delta$. Similarly, for large enough Δ , $q_B(G) \leq \Delta$ for any outerplanar G with maximum degree Δ .*

For planar graphs, the fact that 2Δ colors may be necessary is shown by $K_{2,\Delta}$ since all edges must get a different color in any B-coloring. For outerplanar graphs, it is immediate from Vizing's theorem. However, the constant Δ_0 is needed. Examples can be seen in Fig. 1.

We note that for A-colorings the corresponding problem is still wide open (see [10]).

The purpose of this paper is to prove the conjecture for outerplanar graphs and to give an improved bound for planar graphs. Instead of expressing our results in the manner of Conjecture 1, we instead express it so that, for some function f and constant c , if the number of colors in the palette is at least $\max\{f(\Delta), c\}$ then a B-coloring of the graph with that palette is possible.

First for outerplanar graphs we prove the following, which improves on the bound of $\Delta + 1$ from [7]:

Theorem 2. *Let G be an outerplanar graph with maximum degree at most Δ . Then $q_B(G) \leq \max\{\Delta, 9\}$.*

In Fig. 1(A), there is an outerplanar graph with $\Delta = 4$ that requires 5 colors, so it is an open question as to whether the expression $\max\{\Delta, 9\}$ can be replaced by $\max\{\Delta, 5\}$.

For planar graphs our result is the following, which improves on the bound of $2\Delta + 8$ from [7]:

Theorem 3. *Let G be a planar graph with maximum degree at most Δ . Then $q_B(G) \leq \max\{2\Delta + 4, 22\}$.*

This is still short of the conjectured value of $\max\{2\Delta, c\}$ for some constant c . In Fig. 1(B), there is a planar graph with $\Delta = 4$ that requires 12 colors, so it is open as to whether the value of 22 can be reduced to 12.

A graph G is said to be a *topological copy* of a graph H if G can be formed by subdivisions of the edges of H . If G is a graph with a partial B-coloring of the edges and the edge uv is uncolored, we say that an edge that is incident to u , incident to v or forms a C_4 with uv and receives some color is a *conflict edge*. A color of a conflict edge of uv is a *conflict color*. We say that two edges are *parallel* if they are in the same C_4 but do not share a vertex.

2. Proof of Theorem 2, the outerplanar case

We take note of the following well-known observations of outerplanar graphs:

Lemma 4. *The following properties are true of outerplanar graphs:*

- The property of outerplanarity is hereditary. That is, it is closed under vertex deletion.
- The property of outerplanarity is minor-closed. That is, it is closed under edge-contraction.

Let G be an outerplanar graph on n vertices.

- (a) G has no topological copy of K_4 .
- (b) G has no topological copy of $K_{2,3}$.
- (c) If $n \geq 2$, then $e(G) \leq 2n - 3$.
- (d) G has at least two vertices of degree at most two.
- (e) If G is 2-connected, then G is hamiltonian.
- (f) If G is 2-connected, then either G is a cycle or there is a vertex of degree 4 that has two neighbors with degree 2 or a vertex of degree 3 that has at least one neighbor with degree 2.

Note that if we contract an edge of a planar graph and the edge is in a triangle, then the graph is no longer be simple. Thus we will take care to ensure that any edge we contract will not be in a triangle.

Proof. We will only prove (f), the others can be found e.g. in West's textbook [13]. Let G be a 2-connected outerplanar graph. Let $V_2 \subseteq V = V(G)$ be the set of degree-2 vertices. Observe that for any Hamilton cycle in G , both edges incident to a degree-2 vertex are edges in that Hamilton cycle. If G is not a cycle, then by the fact that outerplanarity is a hereditary property, the graph $G' = G[V - V_2]$ is an outerplanar graph with at least one edge. (Note that G' might not be connected, let alone 2-connected.)

Since G is hamiltonian and every edge incident to a degree-2 vertex belongs to every Hamilton cycle, no vertex is incident to more than two vertices in V_2 . Since G' is outerplanar and not empty, there is a vertex of degree at most 2, but that means that, in G , it either has degree 4 and has two neighbors in V_2 or it has degree 3 and has one or two neighbors in V_2 . $\square$

Lemma 5 below is the main technical lemma for Theorem 2 which gives a bound on the number of conflict edges. The proof is in Section 4.1.

Lemma 5. *In an outerplanar graph G , the number of edges that conflict with the edge xy is at most $d(x) + d(y) + \min\{d(x), d(y), 5\} - 3$.*

If there exists a degree-2 vertex z such that xyz is a triangle, then the number of edges that conflict with the edge xy is at most $d(x) + d(y) + \min\{d(x), d(y), 4\} - 4$.

If the edge xy is not in a C_4 , then the number of edges that conflict with the edge xy is at most $d(x) + d(y) - 2$.

With these important lemmas, we proceed to the proof of the outerplanar result.

Proof of Theorem 2. With foresight, we will set $c = 9$ but use c for convenience and to prevent confusion.

The proof will proceed by induction on $|E(G)|$ but we must be careful. If an edge e is deleted, then a C_4 may be destroyed, which would remove a constraint between parallel edges of this C_4 . Most of the time, we will delete vertices to avoid this complication. The case of $|E(G)| \leq \max\{\Delta, c\}$ is trivial because the edges can be totally multicolored.

Now suppose that any outerplanar graph on fewer than $|E(G)|$ edges with maximum degree at most Δ can be B-colored with at most $\max\{\Delta, c\}$ colors.

First, we may assume that G is 2-connected. Otherwise if G is the result of two graphs G_1 and G_2 that share only a common vertex v , then we can B-color each independently with $\max\{\Delta, c\}$ colors and permute the colors on G_2 so that the edges at v all receive different colors. This results in a B-coloring of G because v cannot be in a C_4 that uses vertices from both $G_1 - v$ and $G_2 - v$.

Fact 6 below results from the fact that if there is a vertex u such that no edges incident to it have at least $\max\{\Delta, c\}$ conflict edges, then we can delete u , color $G - u$ with $\max\{\Delta, c\}$ colors and then greedily color the edges incident to u because there will always be a color remaining.

Fact 6. *We may assume that every vertex u is adjacent to a vertex v such that uv conflicts with at least $\max\{\Delta, c\}$ other edges.*

Fact 7 gives us conditions on which we cannot apply induction. For part (a), we would like to delete an edge uv not in a C_4 , B-color the edges of $G - uv$, and then color uv if a color remains that does not conflict with the previously-colored edges. This can be done unless the sum $d(u) + d(v)$ is too large. For part (b), we would like to contract the edge uv , B-color the edges of the graph that results from contracting the edge uv (which we denote with G/uv), and then color uv if a color remains which does not conflict with the colors on the previously-colored edges. Part (c) is a special case of (b), which provides a better bound when one vertex has degree 2.

Fact 7. *Let $u, v \in V(G)$ and $uv \in E(G)$. We may assume that each of the following occur:*

- (a) *If uv is not in a C_4 , then $d(u) + d(v) \geq \max\{\Delta, c\} + 2$.*
- (b) *If uv is not in a C_3 , then either $d(u) + d(v) \geq \Delta + 3$ or $d(u) + d(v) \geq \max\{\Delta, c\} - 2$.*
- (c) *If uv is not in a C_3 and $d(u) = 2$, then $d(v) \geq \max\{\Delta, c\} - 1$.*
- (d) *Every degree-2 vertex has a neighbor of degree at least $\max\{\Delta, c\} - 1$.*

Proof of Fact 7. (a) By Lemma 5, the number of edges that conflict with uv is at most $d(u) + d(v) - 2$. By induction, we may B-color $G - \{uv\}$ by $\max\{\Delta, c\}$ colors and as long as $d(u) + d(v) - 2 < \max\{\Delta, c\}$, there is a color available for uv . (b) By contracting uv , the fact that outerplanarity is a minor-closed property gives that the resulting graph $G' = G/uv$ is outerplanar and has maximum degree $\max\{\Delta, d(u) + d(v) - 2\}$. Since uv was not in a triangle, G' is a simple graph. Moreover, every pair of edges in G' are that are in conflict in G are also in conflict in G' : If they are adjacent edges in G , they remain adjacent in G' . If they are parallel edges in G , then either they remain parallel in G' or they are in a



Fig. 2. Gadgets for the proof of Theorem 2. Bold edges are outside edges. Dotted edges belong to $G - \{u, v, u'\}$ or $G - \{u, v\}$, respectively.

C_4 which contains uv as an edge in which case, the edges become adjacent in G' . (We note that there could be more constraints in G' than in G , say there could be two edges in some induced C_5 that includes uv that become parallel edges in G' .)

If $d(u) + d(v) - 2 \leq \Delta$, then G' has maximum degree at most Δ , and every pair of edges in $E(G) - \{uv\}$ that conflict in G will also conflict in G' , we may apply induction to G' . By Lemma 5, there are at most $d(u) + d(v) + 5 - 3$ edges that conflict with uv and if $d(u) + d(v) + 2 < \max\{\Delta, c\}$, then there is a color available for uv .

- (c) We follow the same reasoning as in (b) but observe first that if $d(u) = 2$, then $d(u) + d(v) - 2 \leq \Delta$. Again, we apply induction to G' . Then note that since $\min\{d(u), d(v), 5\} = 2$, Lemma 5 gives that there are at most $d(u) + d(v) + 2 - 3 = d(v) + 1$ edges that conflict with uv and if $d(v) + 1 < \max\{\Delta, c\}$, then there is a color available for uv .
- (d) By Lemma 5, if $d(u) = 2$ then for any neighbor v , the number of edges that conflict with uv is at most $d(u) + d(v) + 2 - 3 = d(v) + 1$. By Fact 6, there is a neighbor v such that uv conflicts with at least $\max\{\Delta, c\}$ other edges and so there must be a vertex v adjacent to u such that $d(v) + 1 \geq \max\{\Delta, c\}$. $\square$

With Fact 7 in mind, let us prove the theorem. First, observe that since G is 2-connected, then Lemma 4(e) gives that G is hamiltonian. We call the edges of some fixed Hamilton cycle *outer edges*.

In addition, by Lemma 4(f), there must be at least one structure, which we refer to as a gadget:

- (D_4) a degree-4 vertex v adjacent to two degree-2 vertices u and u' , or
- (D_3) a degree-3 vertex v adjacent to a degree-2 vertex u .

Each of the edges uv and $u'v$ (in the case of D_4) are outer edges.

Consider a D_4 and let w be the neighbor of u where uw is an outer edge and w' be the neighbor of u' where $u'w'$ is an outer edge as illustrated in Fig. 2(A). By Fact 7(c), since $d(v) = 4 < \max\{\Delta, c\} - 1$ ($c \geq 6$ suffices), the edge uv must be in a triangle and the only choice is uvw . So uw (and similarly $u'w'$) must be an edge. Furthermore, by Fact 7(a), since $d(u) + d(v) = 6 < \max\{\Delta, c\} + 2$ ($c \geq 5$ suffices), the edge uv must be in a C_4 . This must be $uww'v$. Furthermore, by Fact 7(d), each of $d(w)$ and $d(w')$ is at least $\max\{\Delta, c\} - 1$.

Now consider D_3 and let w be the neighbor of u and w' be the neighbor of v via an outer edge. The vertices w and w' must be distinct, otherwise G cannot be 2-connected. By Fact 7(c), since $d(v) = 3 < \max\{\Delta, c\} - 1$, the edge uv must be in a triangle and the only choice is uvw . Furthermore, by Fact 7(a), since $d(u) + d(v) = 5 < \max\{\Delta, c\} + 2$, the edge uv must be in a C_4 . This must be $ww'v$.

Thus, Fig. 2(A) and Fig. 2(B) show the gadgets D_4 and D_3 , respectively.

So now we have a strong characterization of the vertices w and w' .

Fact 8. If G is not B -colorable by $\max\{\Delta, c\}$ colors and $c \geq 6$, then $\Delta \geq c$ and the following are true for gadgets:

- (D_4) Both w and w' have degree Δ .
- (D_3) The vertex w has degree Δ and the vertex w' has degree at least $\Delta - 2$.

Proof of Fact 8. First we show that every vertex of degree 2 either has a neighbor of degree Δ or both neighbors have degree $\Delta - 1$. Towards a contradiction, consider a vertex u of degree two with neighbors v_1 and v_2 such that $d(v_1) \geq d(v_2)$ and suppose $d(v_1) \leq \Delta - 1$ and $d(v_2) \leq \Delta - 2$.

By Lemma 5, the edge uv_1 has at most $d(v_1) + 2 + 2 - 3 = d(v_1) + 1$ conflicts, including the edge uv_2 . The edge uv_2 has at most $d(v_2) + 1$ conflicts.

First, B -color the graph $G - u$ with at most $\max\{\Delta, c\}$ colors. Second color uv_1 . There are at most $d(v_1) \leq \Delta - 1 \leq \max\{\Delta, c\} - 1$ colored edges that conflict with uv_1 (since uv_2 is uncolored) and so a remaining color is available for uv_1 . Third color uv_2 . There are at most $d(v_2) + 1 \leq \max\{\Delta, c\} - 1$ colored edges that conflict with uv_2 and so a remaining color is also available for uv_2 . This is a B -coloring of G , a contradiction. Therefore, either $d(v_1) \geq \Delta$ or $d(v_1) = d(v_2) = \Delta - 1$ and observe that if $c > \Delta$, then there is still a color available in both cases.

Turning to D_4 , the above argument implies that, as long as $4 < \max\{\Delta, c\} - 1$ ($c \geq 6$ suffices), it must be the case that $d(w) = d(w') = \Delta$.

Turning to D_3 , the above argument also implies that $d(w) = \Delta$. To bound $d(w')$, we will suppose $d(w') \leq \Delta - 3$. First, B-color $G - \{u, v\}$ (the “ $-$ ” symbol in this case defines the graph G with vertices u and v deleted) with at most $\max\{\Delta, c\}$ colors and color the following edges in order: vw , uw , vw' and uv . We know that $d(u) = 2$, $d(v) = 3$, and $d(w) = \Delta$. Consider Lemma 5.

- vw : The number of edges that conflict with vw is at most $d(v) + d(w) + \min\{d(v), d(w), 4\} - 4 = \Delta + 2$, including the three uncolored edges uw , vw' , uv . Thus, there is a color available for vw .
- uw : The number of edges that conflict with uw is at most $d(u) + d(w) + \min\{d(u), d(w), 5\} - 3 = \Delta + 1$, including the two uncolored edges vw' , uv . Thus, there is a color available for uw .
- vw' : The number of edges that conflict with vw' is at most $d(v) + d(w') + \min\{d(v), d(w'), 5\} - 3 = d(w') + 3 \leq \Delta$, including the uncolored edge uv . Thus, there is a color available for uw .
- uv : The number of edges that conflict with uv is at most $d(u) + d(v) + \min\{d(u), d(v), 5\} - 3 = 4$. Since $\max\{\Delta, c\} \geq 5$, there is a color available for uv .

This concludes the proof of Fact 8. $\square$

So now we have that $\Delta \geq c$ and the set of all w 's and w' 's in a gadget have degree Δ except those vertices that serve only as a w' in gadgets D_3 , which have degree at least $\Delta - 2$.

Now, delete the vertices in each of the gadgets, other than those labeled w and w' . That is, for a D_4 , delete $\{u, v, u'\}$ and for a D_3 delete $\{u, v\}$.

The resulting degrees of the vertices that were in gadgets are at least $\Delta - 2 - 2 = \Delta - 4$ (in the case of D_4 or a w vertex in some D_3) or at least $(\Delta - 2) - 1 - 1 = \Delta - 4$ (in the case of vertices that are only w' vertices in D_3 's). This is because any vertex labeled w or w' can be a part of at most two gadgets and deleting a gadget will delete at most one outer edge from such a vertex.

If $\max\{\Delta, c\} > 8$ (so $c \geq 9$ suffices), then no new vertices of degree at most 4 will be created. But this is a contradiction to Lemma 4(f) because no new gadgets will be created but all the existing ones will be deleted.

Thus, having $\max\{\Delta, 9\}$ colors suffices to B-color an outerplanar graph with maximum degree Δ . $\square$

3. Proof of Theorem 3, the planar case

We take note of the following well-known observations of planar graphs:

Lemma 9. *The following properties are true for planar graphs:*

- *The property of planarity is hereditary. That is, it is closed under vertex deletion.*
- *The property of planarity is minor-closed. That is, it is closed under edge-contraction.*

Let G be a planar graph on n vertices.

- (a) G has no topological copy of K_5 .
- (b) G has no topological copy of $K_{3,3}$.
- (c) If $n \geq 3$, then $e(G) \leq 3n - 6$. Otherwise, $e(G) \leq 3n$.
- (d) If G is bipartite and $n \geq 3$, then $e(G) \leq 2n - 4$. Otherwise if G is bipartite, then $e(G) \leq 2n$.
- (e) The minimum degree of G is at most five.

Lemma 10 below is the main technical lemma for Theorem 3 which gives a bound on the number of conflict edges. The proof is in Section 4.1.1.

Lemma 10. *Let G be a connected planar graph that is not K_2 and let u and v be adjacent vertices.*

If $\min\{\deg(u), \deg(v)\} = 1$, then the number of edges that form a conflict with uv is at most $\max\{\deg(u), \deg(v)\} - 1$. Otherwise, the number of edges that form a conflict with uv is at most:

$$2\deg(u) + 2\deg(v) + \left\lfloor \frac{1}{2} \min\{\deg(u), \deg(v)\} \right\rfloor - 6. \quad (1)$$

With these important lemmas, we proceed to the proof of the planar result.

Proof of Theorem 3. With foresight, we will set $c = 22$ but use c for convenience and to prevent confusion.

In this case, we can proceed by induction on $|V(G)|$. The base case of $\binom{|V(G)|}{2} \leq \max\{2\Delta, c\}$ is trivial because the edges can be totally multicolored.

Now we consider the following theorem of Borodin and Ivanova [2]:

Theorem 11 (Borodin, Ivanova [2]). *Let G be a planar graph with minimum degree $\delta \geq 3$. There is a vertex u of degree $d \geq \delta$ with $d - 2$ neighbors $w_1, \dots, w_{d-2}$ such that one of the following occurs:*

- (i) $d = 3$ and $\deg(w_1) \leq 10$,
- (ii) $d = 4$ and $\deg(w_2) \leq 5$ and $\deg(w_1) \leq 9$,
- (iii) $d = 4$ and $\deg(w_2) = 6$ and $\deg(w_1) \leq 8$,
- (iv) $d = 4$ and $\deg(w_2) = \deg(w_1) = 7$,
- (v) $d = 5$ and $\deg(w_3) = 4$ and $\deg(w_2) = \deg(w_1) = 5$,
- (vi) $d = 5$ and $\deg(w_3) = 5$ and $\deg(w_2) \in \{5, 6\}$ and $\deg(w_1) \in \{5, 6, 7\}$,
- (vii) $\delta = 5$ and $\deg(w_3) = \deg(w_2) = \deg(w_1) = 6$.

Using Lemma 9(e), we see that there is a vertex u of degree at most 5. If $\delta = \delta(G) \leq 2$, then let u be a vertex of minimum degree δ . Otherwise, let u be the vertex given by Theorem 11. Let $d = \deg(u)$.

If $d \geq 3$, then let $N(u) = \{v_1, v_2, w_1, \dots, w_{d-2}\}$. If $\deg(u) = 2$, then let $N(u) = \{v_1, v_2\}$ and if $\deg(u) = 1$, then let $N(u) = \{v\}$. By induction, $G - u$ can be B-colored by $\max\{2\Delta + 4, c\}$ colors.

In each case below, the bound we get on the number of conflicting edges is derived from Lemma 10.

Case 1. $d = 1$.

The number of edges that conflict with uv is at most $\deg(v) - 1 \leq \Delta - 1$. So there are at least $\max\{2\Delta + 4, c\} - (\Delta - 1) \geq \Delta + 5 \geq 1$ colors available for uv .

Case 2. $d = 2$.

The number of edges that conflict with uv_i for $i \in \{1, 2\}$ is at most $2 \cdot 2 + 2\deg(v_i) + \lfloor 2/2 \rfloor - 6 = 2\deg(v_i) - 1 \leq 2\Delta - 1$.

If we color uv_1 first, then since uv_2 is uncolored, there are at least $\max\{2\Delta + 4, c\} - (2\Delta - 1) + 1 \geq 6$ colors available for uv_1 . Then there are at least $\max\{2\Delta + 4, c\} - (2\Delta - 1) \geq 5$ colors available for uv_2 .

Case 3. $d = 3$.

The number of edges that conflict with uv_i for $i \in \{1, 2\}$ is at most $2 \cdot 3 + 2\deg(v_i) + \lfloor 3/2 \rfloor - 6 = 2\deg(v_i) + 1 \leq 2\Delta + 1$.

If we color uv_1 first, then since uv_2 and uw_1 are uncolored, there are at least $\max\{2\Delta + 4, c\} - (2\Delta + 1) + 2 \geq 5$ colors available for uv_1 . Since uw_1 is uncolored, there are at least $\max\{2\Delta + 4, c\} - (2\Delta + 1) + 1 \geq 4$ colors available for uv_2 .

Finally, the number of edges that conflict with uw_1 is at most $2 \cdot 3 + 2\deg(w_1) + \lfloor 3/2 \rfloor - 6 \leq 2\deg(w_1) + 1$. So, there are at least $\max\{2\Delta + 4, c\} - (2\deg(w_1) + 1) \geq \max\{2\Delta + 4, c\} - (2\Delta + 1) \geq 3$ colors available for uw_1 .

Case 4. $d = 4$.

The number of edges that conflict with uv_i for $i \in \{1, 2\}$ is at most $2 \cdot 4 + 2\deg(v_i) + \lfloor 4/2 \rfloor - 6 = 2\deg(v_i) + 4 \leq 2\Delta + 4$.

If we color uv_1 first, then since uv_2 , uw_1 , and uw_2 are uncolored, there are at least $\max\{2\Delta + 4, c\} - (2\Delta + 4) + 3 \geq 3$ colors available for uv_1 . Since uw_1 and uw_2 are uncolored, there are at least $\max\{2\Delta + 4, c\} - (2\Delta + 4) + 2 \geq 2$ colors available for uv_2 .

The number of edges that conflict with uw_1 is at most $2 \cdot 4 + 2\deg(w_1) + \lfloor 4/2 \rfloor - 6 \leq 2\deg(w_1) + 4$. So there are at least $\max\{2\Delta + 4, c\} - (2\deg(w_1) + 4) + 1$ colors available for uw_1 . Similarly, there are at least $\max\{2\Delta + 4, c\} - (2\deg(w_2) + 4)$ colors available for uw_2 .

In the case of Theorem 11(ii), (iii), or (iv), the number of colors available for uw_1 is at least $\max\{2\Delta + 4, c\} - (2\deg(w_1) + 4) + 1 \geq \max\{2\Delta + 4, c\} - (2 \cdot 9 + 4) + 1 \geq \max\{2\Delta + 4, c\} - 21$ and the number of colors available for uw_2 is at least $\max\{2\Delta + 4, c\} - (2\deg(w_2) + 4) \geq \max\{2\Delta + 4, c\} - (2 \cdot 7 + 4) \geq \max\{2\Delta + 4, c\} - 18$. Hence $c \geq 22$ suffices.

Case 5. $d = 5$.

The number of edges that conflict with uv_i for $i \in \{1, 2\}$ is at most $2 \cdot 5 + 2\deg(v_i) + \lfloor 5/2 \rfloor - 6 = 2\deg(v_i) + 6 \leq 2\Delta + 6$.

If we color uv_1 first, then since uv_2 , uw_1 , uw_2 , and uw_3 are uncolored, there are at least $\max\{2\Delta + 4, c\} - (2\Delta + 6) + 4 \geq 2$ colors available for uv_1 . Since uw_1 , uw_2 , and uw_3 are uncolored, there is at least $\max\{2\Delta + 4, c\} - (2\Delta + 6) + 3 \geq 1$ color available for uv_2 .

The number of edges that conflict with uw_1 is at most $2 \cdot 5 + 2\deg(w_1) + \lfloor 5/2 \rfloor - 6 \leq 2\deg(w_1) + 6$. So there are at least $\max\{2\Delta + 4, c\} - (2\deg(w_1) + 6) + 2$ colors available for uw_1 , at least $\max\{2\Delta + 4, c\} - (2\deg(w_2) + 6) + 1$ colors available for uw_2 and at least $\max\{2\Delta + 4, c\} - (2\deg(w_3) + 6)$ colors available for uw_3 .

In the case of Theorem 11(v), (vi), or (vii), the number of colors available for uw_1 is at least $\max\{2\Delta + 4, c\} - (2\deg(w_1) + 6) + 2 \geq \max\{2\Delta + 4, c\} - (2 \cdot 7 + 6) + 2 \geq \max\{2\Delta + 4, c\} - 18$, the number of colors available for uw_2 is at least $\max\{2\Delta + 4, c\} - (2\deg(w_2) + 6) + 1 \geq \max\{2\Delta + 4, c\} - (2 \cdot 6 + 6) + 1 \geq \max\{2\Delta + 4, c\} - 17$, and the number of

colors available for uw_3 is at least $\max\{2\Delta + 4, c\} - (2\deg(w_3) + 6) \geq \max\{2\Delta + 4, c\} - (2 \cdot 6 + 6) \geq \max\{2\Delta + 4, c\} - 18$. Hence $c \geq 19$ suffices.

So in every case, having at least $\max\{2\Delta + 4, 22\}$ colors suffices to B-color the graph G . $\square$

4. Proofs of Lemmas

4.1. Proof of Lemma 5

An edge that conflicts with xy must either be incident to x (there are $\deg(x) - 1$ such edges), incident to y (there are $\deg(y) - 1$ such edges) or must be an edge parallel to uv in some copy of a C_4 .

An edge parallel to xy cannot have both endvertices in $N(x) \cap N(y)$, otherwise that would result in a K_4 , contradicting Lemma 4(a).

An edge parallel to xy should have one endvertex adjacent to x and the other adjacent to y . If z_1z_2 and z_2z_3 are edges that are parallel to xy then it is not possible that both of z_1 and z_3 are neighbors of x or that both of them are neighbors of y . Otherwise, z_2 is adjacent to either y or x respectively and so the graph induced by $\{x, y, z_1, z_2, z_3\}$ would have a copy of $K_{2,3}$, contradicting Lemma 4(b).

So if z_1z_2 and z_2z_3 are edges that are parallel to xy , then without loss of generality, $z_1 \in N(x) \setminus N(y)$, $z_2 \in N(x) \cap N(y)$ and $z_3 \in N(y) \setminus N(x)$. Moreover, for the graph induced by the edges that are parallel to xy , the components are copies of the path P_3 or the path P_2 and by Lemma 4(b) and (a), it can be seen that these must be vertex-disjoint and there are at most two such paths.

Hence, the maximum number of edges parallel to xy is 4 and if there are k parallel edges, then $\deg(x), \deg(y) \geq k + 1$. Thus, the total number of conflict edges is at most $d(x) + d(y) - 2 + \min\{\deg(x) - 1, \deg(y) - 1, 4\} = \deg(x) + \deg(y) + \min\{\deg(x), \deg(y), 5\} - 3$.

If there is a degree-2 common neighbor of x and y , then the maximum number of edges parallel to xy is 2 and if there are k parallel edges, then $d(x), d(y) \geq k + 2$. Thus, if x and y have a degree-2 common neighbor, then the total number of conflict edges is at most $\deg(x) + \deg(y) - 2 + \min\{\deg(x) - 2, \deg(y) - 2, 2\} = \deg(x) + \deg(y) + \min\{\deg(x), \deg(y), 4\} - 4$.

If xy is not in a C_4 , then the edges that conflict with xy are only the ones incident with either x or y , and there are at most $(\deg(x) - 1) + (\deg(y) - 1)$ of such edges. $\square$

4.1.1. Proof of Lemma 10

For adjacent vertices u and v , let

- $W_{uv} = N(u) \cap N(v)$,
- $W_u = N(u) - (N(v) \cup \{v\})$, and
- $W_v = N(v) - (N(u) \cup \{u\})$.

Let G' be the graph consisting of edges not incident to u or v that are conflict edges with uv . That is, these are edges with one endpoint in $W_u \cup W_{uv}$ and the other endpoint in $W_v \cup W_{uv}$. In particular, there are no edges in G' with both endvertices in W_u or with both endvertices in W_v .

Let G'' be the subgraph of G' induced by $W_u \cup W_v$. Note that every edge in G'' has one endpoint in W_u and the other endpoint in W_v .

We will state several facts about G' and G'' that will give a very specific structure to these graphs and will allow us to count the conflict edges.

Fact 12 establishes that there are very few cycles in G' .

Fact 12. *There is no cycle of length at least 5 in G' . Furthermore, every 4-cycle alternates between sets as follows: W_u, W_{uv}, W_v, W_{uv} and every 3-cycle contains either one or two vertices from W_{uv} .*

In particular, G'' is acyclic.

Proof of Fact 12. Suppose that G' has a cycle such that there exist four vertices in cyclic order $x_1 \in W_u \cup W_{uv}$, $y_1 \in W_v \cup W_{uv}$, $x_2 \in W_u \cup W_{uv}$, $y_2 \in W_v \cup W_{uv}$.

This would produce a topological $K_{3,3}$ with one set being $\{x_1, x_2, v\}$ and the other being $\{y_1, y_2, u\}$. Hence, these vertices x_1, y_1, x_2, y_2 cannot exist.

Consider any cycle $v_1, v_2, \dots, v_\ell$ of length $\ell \geq 4$. If each of the vertices is in W_{uv} , then one can find x_1, y_1, x_2, y_2 by choosing them to be consecutive. So, without loss of generality, let $v_3 \in W_u$. If $v_1 \notin W_v$, then since there is no edge in G' with both endvertices in W_u or with both endvertices in W_v we can set $x_1 = v_1$, $y_1 = v_2$, $x_2 = v_3$, and $y_2 = v_4$, a contradiction. So $v_1 \in W_v$.

If $\ell = 4$, then it must be the case that $v_1 \in W_v$, $v_2, v_4 \in W_{uv}$ and $v_3 \in W_u$.

If $\ell \geq 5$ and $v_5 \notin W_v$, then we can set $x_1 = v_5$, $y_1 = v_4$, $x_2 = v_3$, and $y_2 = v_2$, a contradiction.

If $\ell \geq 5$ and $v_5 \in W_v$, then since v_5v_1 cannot be an edge, it is the case that $\ell \geq 6$ and we can set $x_1 = v_6$, $y_1 = v_5$, $x_2 = v_3$, and $y_2 = v_1$, a contradiction.

As far as 3-cycles, since there are no edges with both endvertices in W_u or with both endvertices in W_v , then any 3-cycle must contain at least one vertex from W_{uv} . But if W_{uv} contains a 3-cycle, then adding u and v produces a copy of K_5 , a contradiction.

Since every cycle in G' must contain a member of W_{uv} , the subgraph G'' is acyclic. $\square$

Fact 13 establishes that no vertex has more than two neighbors in W_{uv} , which will be important for our expanded definition of a component in G'' .

Fact 13. Every vertex $x \in V(G')$ has at most two neighbors in W_{uv} .

Proof of Fact 13. If $x \in V(G')$ has three distinct neighbors $z_1, z_2, z_3 \in W_{uv}$, then there is a copy of $K_{3,3}$ with one set being $\{x, u, v\}$ and the other being $\{z_1, z_2, z_3\}$, a contradiction. $\square$

As a result of Fact 12, each connected component of G'' must be a (possibly degenerate) tree. In addition we will define special components.

Definition 14. The vertex set $V(G'') = W_u \cup W_v$ is partitioned as follows:

- Let every nondegenerate connected component be a member of the partition and call it an *ordinary nondegenerate component*.
- If $x \in W_u$ and $y \in W_v$ have two common neighbors in W_{uv} , make $\{x, y\}$ a member of the partition and call it a *special component*.
- Let every $v \in V(G'')$ that is not in either an ordinary nondegenerate component or in a special component form an *ordinary degenerate component* $\{v\}$.

Together, these are simply called the *components* of G'' .

Remark. A special component $\{x, y\}$ may either be a connected component that is a single edge in G'' or a pair of vertices that are isolated in G'' . Taken together with their common neighbors in W_{uv} , say w_1 and w_2 , those four vertices induce at least four edges and possibly five with xy or w_1w_2 . However, both edges cannot be present because, together with u and v , this would form a $K_6 - 2K_2$ which has both a copy of $K_{3,3}$ and a topological copy of K_5 .

It is necessary to establish that components indeed form a partition of $V(G'')$ so that we can analyze each component individually. For any component C , let ∂C denote the neighbors of C in W_{uv} . That is, $\partial C = \{z \in W_{uv} : \exists x \in C \subseteq W_u \cup W_v, xz \in E(G)\}$.

Fact 15. The components of G'' (including the special components) partition $V(G'') = W_u \cup W_v$.

Proof of Fact 15. It suffices to show that a special component has an empty intersection with every other special component and with every ordinary nondegenerate component.

If two special components C_1, C_2 have a nonempty intersection, then $\partial C_1 = \partial C_2$, otherwise some vertex would have three neighbors in W_{uv} , contradicting Fact 13. So, without loss of generality, there is an $x \in W_u$ and distinct $y_1, y_2 \in W_v$ such that x, y_1 , and y_2 have distinct common neighbors $z_1, z_2 \in W_{uv}$. This produces a copy of $K_{3,3}$ with one set being $\{y_1, y_2, u\}$ and the other being $\{z_1, z_2, v\}$. See Fig. 3(A).

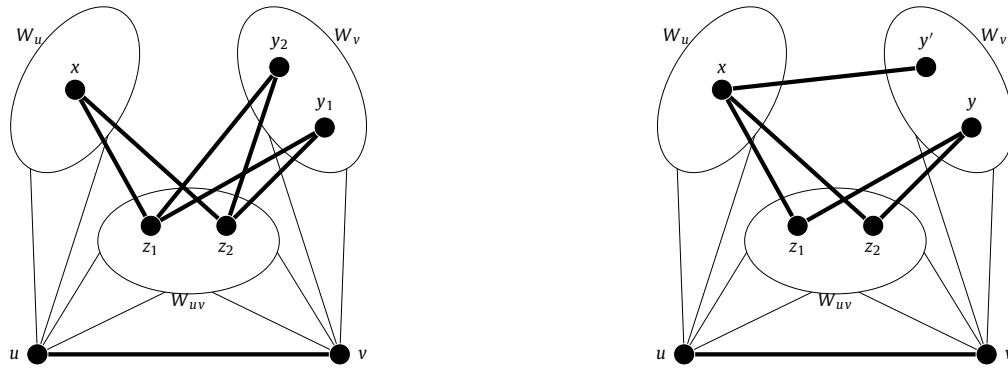
If a special component has a nonempty intersection with an ordinary nondegenerate component, then without loss of generality, there is an $x \in W_u$ and distinct $y, y' \in W_v$ such that x and y have common neighbors $z_1, z_2 \in W_{uv}$ and there is an edge xy' . This produces a topological copy of $K_{3,3}$ with one set being $\{x, y, u\}$ and the other being $\{z_1, z_2, v\}$ (the path $xy'v$ connects x to v). See Fig. 3(B). $\square$

Now we must analyze the components of G'' . Define the following:

- $\mathcal{C}$ is the set of components of G'' .
- $\mathcal{C}_s \subseteq \mathcal{C}$ is the set of special components of G'' .
- $\mathcal{C}_i \subseteq \mathcal{C}$ is the set of ordinary (i.e., not in $\mathcal{C}_s$) components C of G'' for which $|\partial C| = i$.

Fact 16. For every component $C \in \mathcal{C}$, it is the case that $|\partial C| \leq 2$. In particular, $\mathcal{C} = \mathcal{C}_s \cup \mathcal{C}_2 \cup \mathcal{C}_1 \cup \mathcal{C}_0$.

Proof of Fact 16. By Fact 13, no vertex has more than two neighbors in W_{uv} . Consequently, $|\partial C| = 2$ for all $C \in \mathcal{C}_s$.



((A)) Diagram of Fact 15 when two special components have a nonempty intersection.

((B)) Diagram of Fact 15 when one special component has a nonempty intersection with a nontrivial connected component.

Fig. 3. Figures for Fact 15.

Otherwise let $C \in \mathcal{C} - \mathcal{C}_s$, and suppose that there are three (not necessarily distinct) vertices in $V(C)$, called $x_1, x_2, x_3 \in W_u \cup W_v$ and three distinct vertices $z_1, z_2, z_3 \in W_{uv}$ such that $x_i z_i \in E(G)$ for $i = 1, 2, 3$.

We leave it to the reader to verify the following basic graph theory fact:

Fact 17. *If x_1, x_2 , and x_3 are distinct vertices in a connected graph, then either there is a path that contains all three vertices or there is a topological $K_{1,3}$ whose leaves are x_1, x_2 , and x_3 .*

If $\{x_1, x_2, x_3\}$ is a set of size at most 2, then since C is connected, it is trivial to find a topological $K_{1,3}$ with leaves being $\{z_1, z_2, z_3\}$ and all other vertices being in $V(C)$. (The center of that $K_{1,3}$ is the member of $\{x_1, x_2, x_3\}$ with at least two neighbors in $\{z_1, z_2, z_3\}$.) Otherwise, $\{x_1, x_2, x_3\}$ is a set of size 3 and we can apply Fact 17 to find a topological $K_{1,3}$ in C with leaves being $\{x_1, x_2, x_3\}$. In the case of a path, it is a degenerate $K_{1,3}$ in which the center is identified with the leaf that is not an endpoint of the path. Then use the matching $\{x_1 z_1, x_2 z_2, x_3 z_3\}$ to extend it to a topological $K_{1,3}$ with leaves being $\{z_1, z_2, z_3\}$.

If the center of the $K_{1,3}$ is denoted by w , then there is a topological copy of $K_{3,3}$ with one set being $\{z_1, z_2, z_3\}$ and the other set being $\{w, u, v\}$. This is a contradiction to the initial assumption. Thus $|\partial C| \leq 2$ for all $C \in \mathcal{C} - \mathcal{C}_s$ as well. This concludes the proof of Fact 16. $\square$

Now we want to count the number of edges. For any subset of vertices W of G' , denote $e[W] = |E(G'[W])|$. Recall that we ignore any edges with both endpoints in W_u or with both endpoints in W_v . Also, for brevity, denote $\|C\| = |V(C)|$, for any $C \in \mathcal{C}$.

Fact 18. *Let $C \in \mathcal{C}$.*

- If $C \in \mathcal{C}_0$, then $e[C \cup \partial C] = \|C\| - 1$
- For $C \in \mathcal{C}_1$,
 - if $\|C\| = 1$, then $e[C \cup \partial C] = 1 = \|C\|$;
 - if $\|C\| \geq 2$, then $e[C \cup \partial C] \leq \|C\| + 1$.
- For $C \in \mathcal{C}_2$,
 - if $\|C\| = 1$, then $e[C \cup \partial C] \leq 3 = \|C\| + 2$;
 - if $\|C\| = 2$, then $e[C \cup \partial C] \leq 4 = \|C\| + 2$.
 - if $\|C\| \geq 3$, then $e[C \cup \partial C] \leq \|C\| + 3$.
- If $C \in \mathcal{C}_s$, then $e[C \cup \partial C] \leq 5 = \|C\| + 3$.

Proof of Fact 18. We start by making the following observation.

Observation 19. *No vertex in W_{uv} has two neighbors in W_u or two neighbors in W_v that are in the same component.*

To see why the observation is true, without loss of generality, suppose that $z \in W_{uv}$ has two neighbors $y_1, y_2 \in W_v \cap V(C)$ for some component C . Note that $C \notin \mathcal{C}_s$ by definition. Since y_1 and y_2 are in the same connected component, there is a path from y_1 to y_2 in G'' that includes at least one vertex $x \in W_u$. This forms a topological copy of $K_{3,3}$ with one set being $\{y_1, y_2, u\}$ and the other set being $\{x, z, v\}$, a contradiction. See Fig. 4(A). This establishes Observation 19.

With Observation 19 in mind, most of the items become trivial.

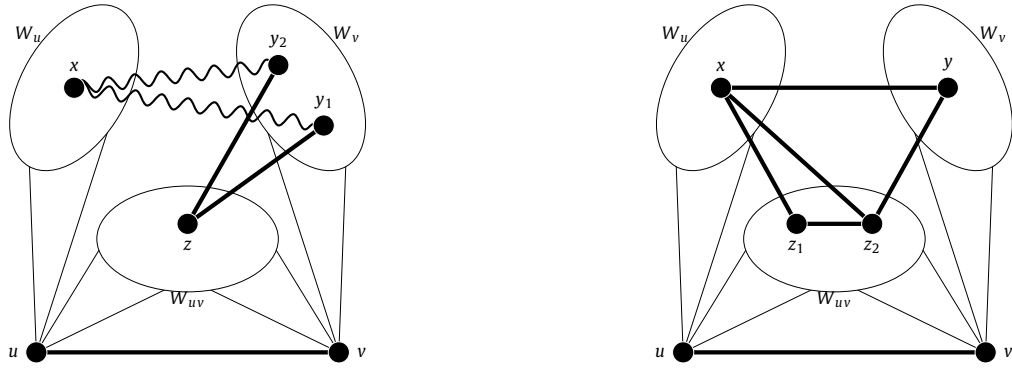
- If $C \in \mathcal{C}_0$, then all of the edges are in C and since C is a tree, the number of edges is precisely $\|C\| - 1$.
- If $C \in \mathcal{C}_1$, then there is exactly one $z \in W_{uv}$ that has neighbors in C . If $\|C\| = 1$, then this is a total of 1 edge, which must exist if $C \in \mathcal{C}_1$. If $\|C\| \geq 2$, then Observation 19 gives that there are at most two edges between C and ∂C , which gives a total of at most $(\|C\| - 1) + 2 = \|C\| + 1$ edges.
- If $C \in \mathcal{C}_2$, then there are exactly two $z_1, z_2 \in W_{uv}$ that have neighbors in C .
 - If $\|C\| = 1$, then there must be two edges between C and ∂C . Together with a possible edge $z_1 z_2$, this gives at most $3 = \|C\| + 2$ edges.
 - If $\|C\| = 2$, then, because $C \in \mathcal{C}_2$, the vertices of C must be $x \in W_u$ and $y \in W_v$ and xy must be an edge. Furthermore, there are at most three edges between C and ∂C , otherwise C would be in $\mathcal{C}_s$. Without loss of generality, suppose the edges between C and ∂C are xz_1, xz_2 and yz_2 . If $z_1 z_2$ is an edge, then this gives a topological K_5 with vertices $\{u, v, z_1, z_2, x\}$ that uses the path xyv , a contradiction. See Fig. 4(B). Hence either $z_1 z_2$ is an edge and there are at most two edges between C and ∂C or $z_1 z_2$ is not an edge and there are at most three edges between C and ∂C . In either case, $e[C \cup \partial C] \leq 1 + 3 = \|C\| + 2$.
 - If $\|C\| \geq 3$, observe that $e[C \cup \partial C] \leq (\|C\| - 1) + 5 = \|C\| + 4$, with equality if and only if $z_1 z_2$ is an edge and each of z_1, z_2 has exactly one neighbor in W_u and exactly one neighbor in W_v . We may, without loss of generality, assume that there are distinct $y_1, y_2 \in W_v$ and $x \in W_u$ such that $xz_1, y_1 z_2, y_2 z_1$ are edges between C and ∂C . By Fact 17, either y_1, x and y_2 are in the same path or there exists a $w \in V(C)$ such that C contains a topological copy of $K_{1,3}$ with w as the center and x, y_1, y_2 as the leaves.
 - * If the order of vertices in the path is y_1, x, y_2 , then there is a topological $K_{3,3}$ with one set being $\{y_1, y_2, u\}$ and the other set being $\{x, z_1, v\}$. See Fig. 5(A).
 - * If the order of vertices in the path is x, y_1, y_2 or y_1, y_2, x , let x' be a vertex in W_u that is along the path from y_1 to y_2 , then there is a topological $K_{3,3}$ with one set being $\{y_1, y_2, u\}$ and the other set being $\{x, x', v\}$. This is because there are paths with vertices in the following order x, z_1, y_1 and x, z_1, z_2, y_2 , as well as edges between the rest of the pairs.
 - * If there is a star centered at w , then there is a topological $K_{3,3}$ with one set being $\{x, y_1, y_2\}$ and the other set being $\{w, z_1, v\}$. This is because there are paths with endpoints w, x , and w, y_1 and w, y_2 , as well as edges between the rest of the pairs.

Since each case is a contradiction, $e[C \cup \partial C] \leq \|C\| + 3$, as claimed.

- If $C \in \mathcal{C}_s$, then $C \cup \partial C$ spans 4 vertices. Let $C = \{x, y\}$ with $x \in W_u$ and $y \in W_v$. If $e[C \cup \partial C] = 6$ then the graph induced by $C \cup \partial C \cup \{u, v\}$ induces a copy of $K_6 - 2K_2$. That is, it is a complete graph on 6 vertices with two nonadjacent edges missing (specifically uy and xv). See Fig. 5(B). Such a graph has a copy of $K_{3,3}$ (and a topological copy of K_5), a contradiction. Thus, there are at most $5 = \|C\| + 3$ edges induced by $C \cup \partial C$. $\square$

Now we are ready to complete the proof of Lemma 10. First, we will require a bit more notation:

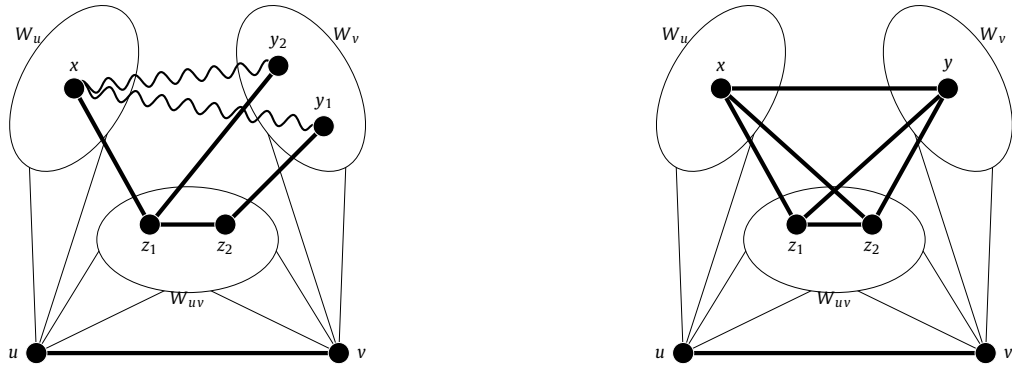
- Define a partition $\mathcal{C}_1 = \mathcal{C}'_1(u) \cup \mathcal{C}'_1(v) \cup \mathcal{C}''_1$; such that
 - for $\alpha \in \{u, v\}$, let $\mathcal{C}'_1(\alpha)$ denote the set of $C \in \mathcal{C}_1$ with $\|C\| = 1$ such that its vertex is in W_α ,
 - let $\mathcal{C}''_1$ denote the set of $C \in \mathcal{C}_1$ with $\|C\| \geq 2$.
- Define a partition $\mathcal{C}_2 = \mathcal{C}'_2(u) \cup \mathcal{C}'_2(v) \cup \mathcal{C}''_2 \cup \mathcal{C}'''_2$; such that
 - for $\alpha \in \{u, v\}$, let $\mathcal{C}'_2(\alpha)$ denote the set of $C \in \mathcal{C}_2$ with $\|C\| = 1$ such that its vertex is in W_α ,
 - let $\mathcal{C}''_2$ denote the set of $C \in \mathcal{C}_2$ with $\|C\| = 2$, and
 - let $\mathcal{C}'''_2$ denote the set of $C \in \mathcal{C}_2$ with $\|C\| \geq 3$.
- Let $\mathcal{K}_2$ denote the set of edges $z_1 z_2$ in $G[W_{uv}]$ such that there is no $C \in \mathcal{C}_2 \cup \mathcal{C}_s$ for which $\partial C = \{z_1, z_2\}$. These are the only edges not accounted for by considering $C \cup \partial C$ for some $C \in \mathcal{C}$.



((A)) Diagram of Observation 19. The wavy line represents a path that begins at y_1 , ends at y_2 , contains x and alternates between W_v and W_u .

((B)) Diagram of Fact 18 in the case where $C \in \mathcal{C}_2$, $\|C\| = 2$ and there is an edge z_1z_2 .

Fig. 4. Figures for Fact 18.



((A)) Diagram of Fact 18 in the case where $C \in \mathcal{C}_2$, $\|C\| \geq 3$ and y_1 , x , and y_2 are in the same path. The wavy line represents a path that begins at y_1 , ends at y_2 , contains x and alternates between W_v and W_u .

((B)) Diagram of Fact 18 in the case where $C \in \mathcal{C}_s$ and $e[C \cup \partial C] = 6$.

Fig. 5. Figures for Fact 18.

By Fact 18, the total number of edges that conflict with uv is

$$\begin{aligned}
 & (\deg(u) - 1) + (\deg(v) - 1) + |\mathcal{K}_2| + \sum_{C \in \mathcal{C}} e[C \cup \partial C] \\
 & \leq (|W_u| + |W_{uv}|) + (|W_v| + |W_{uv}|) + |\mathcal{K}_2| + \sum_{C \in \mathcal{C}} \|C\| \\
 & \quad - |C_0| + |C'_1| + 2|C'_2(u)| + 2|C'_2(v)| + 2|C''_2| + 3|C'''_2| + 3|C_s| \\
 & = 2|W_u| + 2|W_v| + 2|W_{uv}| - |C_0| \\
 & \quad + |\mathcal{K}_2| + |C'_1| + 2|C'_2(u)| + 2|C'_2(v)| + 2|C''_2| + 3|C'''_2| + 3|C_s|. \tag{2}
 \end{aligned}$$

Now we consider our two different conclusions, the general bound and the bound in which $\deg(u)$ and the degrees of its neighbors other than v are small.

If $|W_{uv}| = 0$ and $\min\{\deg(u), \deg(v)\} \geq 2$, then the number of conflict edges is at most

$$2|W_u| + 2|W_v| - |C_0| \leq 2\deg(u) + 2\deg(v) - 5.$$

Otherwise, we need an inequality to bound the number of members of $\mathcal{C}_2 \cup \mathcal{C}_s$.

Fact 20. Let $|W_{uv}| \geq 1$.

$$|C'_1(u)| + |C''_1| + |C'_2(u)| + |C''_2| + |C'''_2| + |C_s| \leq |W_u|, \quad (3)$$

$$|C'_1(v)| + |C''_1| + |C'_2(v)| + |C''_2| + |C'''_2| + |C_s| \leq |W_v|, \quad (4)$$

$$2|K_2| + |C'_1| + 2|C_2| + 2|C_s| \leq 2|W_{uv}|, \quad (5)$$

$$|K_2| + |C_2| + |C_s| \leq |W_{uv}| - 1. \quad (6)$$

Proof of Fact 20. The case of inequality (3) simply follows from the fact that the components of the type mentioned on the left-hand side of (3) partition a subset of W_u . The case of inequality (4) follows similarly when applied to W_v .

For inequalities (5) and (6), we create an auxiliary looped graph Γ such that $V(\Gamma) = W_{uv}$ and edges as follows: (a) For any distinct $z_1, z_2 \in V(\Gamma)$, there is an edge $z_1z_2 \in E(\Gamma)$ if and only if either there is a $C \in \mathcal{C}_2 \cup \mathcal{C}_s$ such that $\partial C = \{z_1, z_2\}$ or $z_1z_2 \in K_2$. (b) The vertex z has a loop for every $C \in \mathcal{C}_1$ such that $\partial C = \{z\}$.

The case of (5) will follow from showing that the maximum degree of Γ is at most 2, where loops count once. Let z have degree 3 and for each edge incident to z , including loops, denote the vertices x_i and y_i as follows:

- If there exists a $z_i \in W_{uv} - \{z\}$ such that zz_i is an edge in K_2 , then let $x_i = z_i$ and $y_i = z_i$ also.
- If $C \in \mathcal{C}'_1$ satisfies $\partial C = \{z\}$, then let x_i be a neighbor of z in $(W_u \cup W_v) \cap C$. If $x_i \in W_u$ then let y_i be a neighbor of x_i in W_v . If $x_i \in W_v$ then let y_i be a neighbor of x_i in W_u .
- If $C \in \mathcal{C}_2$ satisfies $\partial C = \{z, z_i\}$, then there is a path from the neighbor of z in C (let this neighbor be x_i) to z_i (let $y_i = z_i$).
- If $C \in \mathcal{C}_s$ satisfies $\partial C = \{z, z_i\}$, then $C \cup \partial C$ has a path zx_iz_i , where $x_i \in W_u$. Let $y_i = z_i$.

This produces a topological copy of $K_{3,3}$ with sets $\{z, u, v\}$ and $\{x_1, x_2, x_3\}$. For $i = 1, 2, 3$, there are edges zx_i . In addition, either there is an edge ux_i and a vx_i path with y_i as an intermediate vertex or an edge vx_i and a ux_i path with y_i as an intermediate vertex. Each such path is internally disjoint. This produces a topological $K_{3,3}$ and is a contradiction. Therefore, every vertex in Γ has degree at most two and because the sum of the degrees is at most $2|V(\Gamma)| = 2|W_{uv}|$, we have established (5).

As for inequality (6), we want to show that Γ has no cycles (other than the loops). Indeed, for every edge $z_1z_2 \in E(\Gamma)$, there is either an edge $z_1z_2 \in E(G)$ or a distinct component C such that there is a path from z_1 to z_2 such that all of its internal vertices are members of C . So if Γ has a non-loop cycle, then G' has a cycle with at least three vertices in W_{uv} . However, Fact 12 gives that G' can have no cycles with three vertices in W_{uv} .

Thus Γ has no non-loop cycles and if $E(\Gamma)$ denotes the non-loop edge set, then $|E(\Gamma)| \leq |V(\Gamma)| - 1 = |W_{uv}| - 1$, establishing (6). $\square$

Now we use Fact 20 to bound the expression in (2). Recall that if $|W_{uv}| = 0$ and $\min\{\deg(u), \deg(v)\} \geq 2$, then the number of conflict edges is at most $2\deg(u) + 2\deg(v) - 5$. Otherwise, there are three bounds we use for the number of conflict edges:

$$(2) \leq 2|W_u| + 2|W_v| + 2|W_{uv}| + \min \left\{ \begin{array}{l} 3|W_{uv}| - 1, \\ \min\{|W_u|, |W_v|\} + 2(|W_{uv}| - 1), \\ (|W_u| + |W_v|) + (|W_{uv}| - 1) \end{array} \right\}.$$

The first comes from adding (5) + (6). The second comes from adding $\min\{(3), (4)\} + 2 \cdot (6)$. The third comes from adding (3) + (4) + (6). Finally, the minimum is taken over the right-hand side.

Now we find an upper bound in terms of $\deg(u) = |W_u| + |W_{uv}| + 1$, $\deg(v) = |W_v| + |W_{uv}| + 1$, and $\deg(u, v) = |W_{uv}|$.

$$(2) \leq 2\deg(u) + 2\deg(v) - 6 + \min \left\{ \begin{array}{l} \deg(u, v) + 1, \\ \min\{\deg(u), \deg(v)\} - \deg(u, v) - 1, \\ \deg(u) + \deg(v) - 3\deg(u, v) - 1 \end{array} \right\}. \quad (7)$$

Let $m(u, v) = \min\{\deg(u), \deg(v)\}$ and $M(u, v) = \max\{\deg(u), \deg(v)\}$. If $2\deg(u, v) \leq m(u, v) - 2$, then

$$\deg(u, v) + 1 \leq \frac{1}{2}m(u, v).$$

If $m(u, v) - 1 \leq 2\deg(u, v) \leq M(u, v) - 1$, then by substituting the lower bound for $\deg(u, v)$,

$$\min\{\deg(u), \deg(v)\} - \deg(u, v) - 1 \leq \frac{1}{2}m(u, v) - \frac{1}{2}.$$

If $M(u, v) \leq 2\deg(u, v)$, then

$$\begin{aligned}
\deg(u) + \deg(v) - 3 \deg(u, v) - 1 &\leq m(u, v) + \mathfrak{M}(u, v) - 3\left(\frac{1}{2}\mathfrak{M}(u, v)\right) - 1 \\
&\leq \frac{1}{2}m(u, v) + \frac{1}{2}(m(u, v) - \mathfrak{M}(u, v)) - 1 \\
&\leq \frac{1}{2}m(u, v) - 1.
\end{aligned}$$

Note that the upper bound of $\frac{1}{2}m(u, v)$ we obtain for (7) is consistent with the case of $|W_{uv}| = \deg(u, v) = 0$ and $m(u, v) \geq 2$, which yields an upper bound of $2 \deg(u) + 2 \deg(v) - 5$. Since (2) must be an integer, the expression gives the desired general bound

$$(2) \leq 2 \deg(u) + 2 \deg(v) - 6 + \left\lfloor \frac{1}{2} \min \{ \deg(u), \deg(v) \} \right\rfloor. \quad \square$$

5. Concluding remarks

In the proof of Theorem 2, there is only one place which requires $c > 7$ and that is the very end of the proof. Thus, it may be possible to improve the constant term for outerplanar graphs to $\max\{\Delta, c\}$ for some $c < 9$.

We note that a more specific lower bound for $q_B(G)$ is $\max\{\Delta, 2\Delta_2\}$, where Δ_2 is the largest t for which G contains a subgraph $K_{2,t}$. We call Δ_2 the *maximum common degree* in this case. Is it possible that there is a constant c such that for any planar graph G with maximum degree Δ and maximum common degree Δ_2 , $q_B(G) \leq \max\{\Delta, 2\Delta_2, c\}$?

We could reduce the bound for planar graphs in Theorem 3 to an expression of the form $\max\{2\Delta + 3, c\}$ if we could affirmatively resolve the following question:

Question 21. Does there exist an absolute constant c such that for every planar graph with minimum degree $\delta \geq 3$, there exists one of the following:

- a vertex of degree 3 with a neighbor of degree at most c ,
- a vertex of degree 4 with two neighbors, each of degree at most c , or
- a vertex of degree 5 with four neighbors, each of degree at most c ?

We note that it is not possible to replace the second bullet point to “a vertex of degree 4 with three neighbors, each of degree at most c ” because the double-wheel would contradict that. In the double-wheel, there are two vertices of degree $\Delta = n - 2$ and $n - 2$ vertices of degree 4 such that each degree-4 vertex is adjacent to both degree- Δ vertices.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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